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AMBIGUITY OF THE NEGATIVE SIGN

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The goal of this study was to understand errors that student make when simplifying exponential expressions. College students enrolled in four college mathematics courses were asked to simplify and compare such expressions. Quantitative analysis identified three persistent errors: interpreting negative bases, negative exponents, and parentheses. Qualitative methods were used to examine why they made these errors. Analysis indicated that students frequently misinterpret the negative sign when attached to the base of an exponential expression. We theorize that students' concept image of the negative sign must move beyond rule models to correctly interpret numbers in exponential expressions.

Students entering post-secondary schools often need frequent reminders on how to use algebraic properties while solving problems in pre-calculus and calculus classes. Algebra provides the foundation to advanced mathematical thinking; and proficiency in algebraic manipulations is essential for students if they want to enter STEM careers (Liston & O'Donoghue, 2010). The terminology and rules of algebra contain little meaning to many students and appear arbitrary (Demby, 1997; Kieran, 2007). Rules are memorized and are considered to be similar to the rules of a game. Many students have difficulty keeping track of memorized rules that appear to be contradictory at times.

Interpreting the equal sign is one example that has been studied extensively. For example, Knuth et al. (2006) found that an operational conception of the equal sign, the idea that the equal sign was a signal to do something, was more commonly held among middle school students than a relational conception of the equal sign. Students with a relational conception understood the equal sign as representing congruence between the expressions on either side of the equal sign. They found middle school students with the relational understanding were much more successful at solving algebraic equations. Thus, the misconception of the equal sign as a prompt to do something interfered with students' mathematical development.

Carraher and Schliemann (2007) described the difficulties students have bridging arithmetic to algebra and in particular interpreting mathematical symbols. Kieran (2007) expanded those notions by further discussing the development of algebraic thinking in middle and high school. She noted that considerable research exists that describes the ways in which students use variables, expressions, and equations. However, there is relatively little research on students' interpretation of the negative sign which is interpreted as either an operation (subtract) or a negative number. This research paper is part of an on-going study in which the authors are examining pedagogical approaches and interventions that increase precalculus students' achievement. Here we investigate how college students interpret a negative sign that is embedded in exponential expressions.

Previous Research

Students must overcome numerous obstacles to become fluent in algebra, including the interpretation of operations implied by the positioning of symbols next to each other (Lee & Messner, 2000). Research (Chalouh & Herscovics, 1983) on concatenations indicated that many students had difficulty interpreting both mixed numbers where addition is implied or algebraic expressions of the form *3a.* After many years of high school experiences learning that multiplication was implied in concatenations (*3a*), some college students applied multiplication with mixed numbers. They simplified $3\,\frac{2}{3}$ to be $3 \times \frac{2}{3} = 2$. Clearly, these students over generalized one set of meaning that is appropriate for expressions like *3a.* Matz (1980) noted that

many students did not know how to interpret a negative sign and exponents. For example, students frequently confuse whether -3^2 is 9 or -9.

Vlassis (2004) examined how middle school students interpreted negativity. The eighth-grade students conceptualized negativity as a process, linked to the binary operation of subtraction. Negative nine made sense to these students, for example, in an expression such as *n* – 9 but – 9 was more difficult to explain. She concluded that the different uses of the negative sign are counterintuitive and an obstacle for students to overcome.

Teachers often use a textbook to guide their instruction; Lee and Messner (2000) examined opportunities for students to work with the negative sign in sixth- through ninth-grade mathematics curricula. Lee and Messner found that textbooks typically provide multiple opportunities to work with positive and negative integers but seldom offer experiences with negative fractions, exponents, decimals, and mixed numbers. To support students' understanding of – *n*, some textbooks use colored chips in which red is negative and black is positive (wikiversity, 2010). For example, to combine – 9 and +4, students take nine red chips and four black chips. The four black chips annihilate four of the red chips with the result of five red chips, negative five. Here, students develop the notion that the negative sign indicates an entity that neutralizes a positive chip with the net result of zero.

We theorize that students encounter the notion of additive inverse but it is not always clearly articulated or connected to the idea of an identity. Without discussion on the different interpretations of the negative sign, many students have difficulty interpreting the negative sign in concatenations. Kieran (2007) suggests that additional research is needed to understand the reasons for this difficulty.

Theoretical Framework

A mathematical concept can be thought of as a complex web of ideas constructed by each individual from mathematical definitions and mental constructs (Tall & Vinner, 1981; Vinner, 1992). Tall and Vinner described these two components as concept definition and concept image. A concept definition is a set of words that is used to specify the concept. The definition may be phrased in language accepted by the mathematical community or in everyday language taught by teachers who create short mnemonics to help students remember procedures.

For example, teachers and textbooks often use the phrase *opposite of* rather than the mathematical term *additive inverse* to describe the relationship between a number, its negative and the additive identity, zero. A concept image includes "all of the mental pictures and associated properties and processes." (Tall & Vinner, 1981, p. 152). For additive inverse, the image may include $4 + (-4) = 0$ as (a) four jumps to the right and then 4 jumps to the left on a number line or (b) two numbers positioned equidistant but on the opposite side of zero on a number line. Thus, the concept of additive inverse is a synthesis of the concept definition and images associated with it. Vinner (1992) suggested that individuals create idiosyncratic images that may interfere with the development of new concepts.

Concepts develop over time and are important for advanced mathematical thinking (Tall, 1991). They allow for generalization, inference, different levels of abstraction, cognitive economy, and communication (Palmeri & Noelle, 2003). Palmeri and Noelle theorized that a concept image becomes more sophisticated through three stages: rule model, prototype model, and exemplar model. During the stage of the *rule model*, conceptual rules are carefully worded definitions that are internalized by students and used to carry out procedures or to determine whether an object is a member of a set. The *prototype model* indicates the stage in which an

individual recognizes nuances that distinguish individuality among the members of a set. For example, if we consider the set of all triangles, members may look quite different with different side lengths and angles; however, all have exactly three sides. These images become prototypes that allow the individual to check whether an object is a member of the set. Through the prototype model, the concept image becomes more sophisticated and includes both a concept definition and a variety of concept images that illustrate diversity. The *exemplar model* enables individuals to synthesize information about both the commonality and variability of category exemplars and indicates a structural understanding of the concept. Through the exemplar model, the concept image becomes yet more complex and the concept definitions and images are intertwined. For example, students in this stage might recognize how changing the coefficients of a function changes features of its graph. The individual compares the symbolic representation with prototype images and manipulates the images in such a way to describe the effect of the new coefficients. The individual is aware of the nuances between different members and can modify them at will through an understanding of the underlying mathematical structure.

Methods

For this study, we assessed 904 students enrolled in college algebra, pre-calculus, and first and second semester calculus for their ability to simplify and compare exponential expressions. The assessment contained 19 problems involving exponential expressions. Quantitative methods were used to identify the problems that a large number of students missed across the four class levels. There were four problems in which more than 40% of students at all levels solved incorrectly. We labeled these problems as indicators of persistent errors.

To understand why students consistently made these errors, we sought to interview 50 students. However, only 19 students agreed to the interview with a $10 stipend paid for the half-hour interview. These 19 students were enrolled in college algebra, pre-calculus or calculus 2. During the interview, students were asked to read and resolve each of the four problems and explain their reasoning aloud as they solved them. Questions were asked to help identify the concept definitions and images that students used while solving the problems. Conceptual cross-case matrix analysis (Miles & Huberman, 1994) was used to characterize students' explanations to identify commonalities among student explanations.

Findings

Through quantitative and qualitative analysis, three persistent errors emerged around the following concepts: interpretation of a negative sign preceding a base; the role of parentheses in an exponential expression; interpretation of a negative exponent. This paper is based on our qualitative analysis of the first two errors which are related. Students' difficulty distinguishing between a negative number raised to a power and the additive inverse of a number raised to a power became evident by examining their explanations of three problems. Two of the problems required students to simplify exponential expressions and the third problem involved comparing two exponential expressions.

Interpretation of a Negative Sign Preceding a Base

Two problems on the assessment required students to simplify an expression where either a negative sign preceded the base or was part of the base. One asked students to simplify $(-8)^{2/3}$ and the second to simplify $-9^{3/2}$. All of the students who incorrectly simplified the first problem read, "Negative eight raised to the $\frac{2}{3}$ power." Most of the students who incorrectly simplified the

second problem read, "Negative nine raised to the $\frac{3}{2}$ power." Other variations of the second problem were, "I'm not sure how to read that [pointing to $-9^{3/2}$]. It's different from that [pointing to $(-8)^{2/3}$] because it has parenthesis." and "I never know whether the negative sign is included." The reading of these two problems with a negative sign was similar. Students read both problems as if the negative sign was part of the base. They did not differentiate when the negative sign indicated the additive inverse or a negative number. Only one student who misread the expression proceeded to solve the problem correctly. She wrote on her paper, "$-9^{3/2} = -\sqrt{9^3} = -\sqrt{9 \times 9 \times 9} = -\sqrt{81 \times 9} = -\sqrt{729}$." She did not finish simplifying the expression because she "always used a calculator." It is interesting to note that she did not find the square root of nine before cubing it. We interpreted her solution strategy as a reliance on rules, first raise the base to the indicated exponent then find the root of the number.

All of the other students who misread $-9^{3/2}$ and attempted to solve it made the same error. They interpreted the negative sign as attached to the nine before any power was taken. They wrote "$\sqrt{-9^3}$" or stated that the answer was an imaginary number and could not be solved. Four students further simplified $\sqrt{-9^3}$ and their answers resulted in: $3i$, $i\sqrt{729}$, or $(3i)^2 = -81i$. Clearly, they made additional errors in simplifying the expression; however, the error we were most interested in was the initial one of interpreting $-9^{3/2}$ as $(-9)^{3/2}$ rather than the additive inverse of $9^{3/2}$.

Two students correctly interpreted the expression, $-9^{3/2}$. One pre-calculus student correctly stated the steps to follow to simplify it, "Take nine then cube it, then square root it, then make it negative." She read the problem by translating it into the procedure to simplify the problem rather than the more precise reading that a calculus student made, "Take the negative of nine to the $\frac{3}{2}$ power." Both of these students recorded the correct answer.

Only one student we interviewed recognized that taking the square root first and then cubing the result would make the computation easier. It appears most students learned an order of operation that begins with applying the power indicated by the numerator in the exponent followed by computing the root indicated by the denominator in the exponent regardless of the efficiency of this order.

Role of Parentheses in Exponential Expressions

The assessment required students to compare two expressions, $(-17)^8$ and -17^8. Five of the 19 students interviewed had a correct answer and gave a similar explanation, "Negative 17 to the eighth [pointing at $(-17)^8$] is larger than negative 17 raised to the eighth power [pointing at -17^8] because the [even] power of a negative number is positive. Negative 17 to the eighth [pointing at $(-17)^8$] is a positive number and larger than negative 17 to the eighth [pointing at -17^8]."

In contrast, the students who were incorrect read both expressions as "negative 17 to the eighth" claimed that these two numbers were equal. Six students stated, "Parentheses don't matter." When probed, they explained that parentheses matter only when there are more terms. With additional terms, the parentheses indicate what operations should be completed first. One student reasoned, "Parentheses tell you the order of operation, what needs to be done first. You only need them for something like $4(17 - 9)$. They tell you that you have to subtract before multiplying." The other students who gave incorrect answers explained that both expressions were raised to an even exponent and thus both were positive and that –17 and 17 were the same

distance from zero, thus the same. Clearly, the concept image of $(-17)^8$ and -17^8 were identical because –17 and (–17) had identical meaning.

The probing questions that we asked about whether the two numbers were the same prompted one calculus student to retrieve a kernel of forgotten mathematics. Reflecting, he explained that -17^8 could be thought of as -1×17^8. Thus it was a negative number and smaller than the other expression. Immediately after completing this problem, he said, "I made a mistake. I don't know why I did that. I saw the negative sign in that problem [pointing to $-9^{3/2}$] and the square root and automatically wrote it off." At this point, he went back and correctly simplified the previous expression, $-9^{3/2}$. We suspect that the reason for his mistake was not an oversight, rather it indicated an important misconception that was linked to his concept image of negative numbers. The negative sign was irrevocably linked to a number and was interpreted as an indicator of a negative number unless it followed a number or variable.

Discussion

Consistent with findings of previous researchers (Chalouh & Herscovics, 1983; Lee & Messner, 2000; Matz 1980), college students in this study had difficulty interpreting the negative sign in concatenations. Many students did not distinguish a negative number raised to a power from the additive inverse of a number raised to a power. Their concept image was limited to a rule model in which they carried out a procedure to interpret any number with a negative sign, regardless of other mathematical symbols that were next to it. The concept image of $-n$ was limited to a negative number and its meaning did not change when it was raised to an exponential power.

Parenthetically, none of the students that we interviewed used the language of additive inverse when discussing problems on the assessment. Conceptualizing -9 as an additive inverse of 9 suggests a relational interpretation; however, students frequently failed to extend this relational understanding to more complex expressions. In particular, they failed to recognize the symbol $-9^{3/2}$ as the additive inverse of $9^{3/2}$. However, in an operational context such as $1 - 9^{3/2}$, students recognized that $9^{3/2}$ was to be subtracted from the number one. In this mathematical context, students first simplified the exponential expression then found the difference.

When asked if there is an inconsistency between these two interpretations, these college students explained that "…the minus sign means to minus when a number precedes it." Here it is clear that many students may not think of the negative sign as indicating an additive inverse that can be applied to more complex expressions. Without recognizing the mathematical meaning of an additive inverse that is linked to the concept of an additive identity, these students did not develop a more sophisticated concept image of the negative sign. In addition, this may explain why students do not recognize n^{-1} as the multiplicative inverse of n.

Removed from a procedural context, students tend to misinterpret the symbol $-9^{3/2}$. Negative numbers become mathematical objects that are located on the number line. The associated concept image of a negative number is a dot on the number line to the left of zero, equidistant from zero as its positive counterpart. Students fail to separate the negative sign and the base, it becomes an inseparable object. This was made explicit by several of the students as they explained the meaning of -17^8 and $(-17)^8$. Students interpreted both expressions as a negative number that is raised to the power 8. To them, the parentheses were unnecessary because the negative sign and 17 are combined to form an inseparable object, -17. They correctly recognized -17 and (-17) as the same mathematical object but erroneously extended this to -17^8 and $(-17)^8$. The rule for interpreting a negative sign was resistant to recognizing

that its meaning can change when other mathematical symbols are added.

The concept image that these college students developed was different from the eighth-grade students in Vlassis's (2004) study. Vlassis found that the eighth-grade students interpreted negativity as the operation of subtraction. They saw the negative sign as an indication to conduct a procedure. In contrast, the college students in our study interpreted negativity as a mathematical object or entity with a distinct location on a number line. These students associated the negative sign as an operation when it was preceded by a number. In this case they interpreted it differently. Thus, in one sense, these college students were moving toward a concept image with two definitions. We theorize that simultaneous recognition of these two concept rules would lead to a prototype model in which students recognize similarities and differences in the concept images. Thus, they could select the appropriate concept image for a given context. Integration of these images would lead to an exemplar model and students could flexible apply these images in different concatenations. Students using an exemplar model would understand the function of the negative sign to indicate the multiplicative or additive inverse of a number and extent this meaning to more complex situations.

We theorize that students' concept image of negative numbers is not sufficiently well developed to enable them to recognize the negative sign as a relational concept (additive inverse) when the number in question involves an exponent. However, if the number is embedded in an operational context, they generally interpret it correctly.

From our experience, this problem continues to show up in advanced mathematics classes. Students interpret $-x$ as a negative number regardless of the value of x. For example, if students are told that (a, b) is a point in Quadrant II then asked, "In which quadrant does the point $(-a, b)$ lie?" they fail to interpret $-a$ as a positive number. We theorize that it is critical for students to develop a more sophisticated concept image of $-x$ that includes the concept definitions of a negative number and the additive inverse to correctly interpret the negative sign in concatenations.

Implications

The goal of this research project was to understand the fundamental reasons why students have difficulty simplifying exponential expressions. We found that a limited concept image of a negative number was a confounding factor. This inflexible image led students to misinterpret the negative sign in concatenations. We suggest that teachers need to introduce the language of "additive inverse" and "multiplicative inverse" into the class room to help student develop a more sophisticated understanding of the negative sign. In addition, students need more time investigating the interpretation of the negative sign in different concatenations.

We found it interesting that eighth-grade students in Vlassis's (2004) study created a concept image of negativity as an operation. As students mature this image seemed to be replaced by negative numbers as entities unless it was preceded by another mathematical expression . Additional research is needed to understand how students' concept of negativity develops over time. The goal is to help students create more sophisticated concept images that are based on exemplar models rather than rule-based models. We agree that colored chips can be helpful to students as they investigate the meaning of the negative sign but over reliance on one interpretation of the negative sign can hinder students' development of a more sophisticated concept image. They need to spend more time on different interpretations of the negative sign in concatenations. This can be accomplished by creating mathematical contexts that produce cognitive conflict forcing students to reexamine their prototypes for consistency. Continued

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research is needed to better understand the reasons behind persistent errors that students make when interpreting the negative sign in this and other contexts.

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